RAYLEIGH WAVES IN A NONHOMOGENEOUS ELASTIC SLAB

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We examine the problem of the characteristic oscillations of an isotropic elastic slab with plane parallel boundaries and with Lamé parameters and a density which are smooth functions of the vertical coordinate. The dispersion equations for the phase velocity are obtained, and an asymptotic investigation of these equations is carried out for large wave numbers. It is shown that the form of the family of dispersion curves essentially depends on whether or not the Rayleigh velocity exceeds the smallest value of the velocity of a transverse wave in the slab. As a preliminary in this investigation we construct the asymptotic of the solutions of a certain system of ordinary differential equations on an interval which contains a turning point.

1. Construction of the solution. We consider a slab $a \leqslant z \leqslant b$, in which the displacenent vector $\mathbf{u}(x, y, z, t) = (u_x, u_y, u_z)$ satisfies the system of elastic equations

$$L\mathbf{u} \equiv (\lambda + \mu) \nabla (\nabla, \mathbf{u}) + \mu \triangle \mathbf{u} + (\nabla, \mathbf{u}) \nabla \lambda + \\ + (\nabla\mu, \nabla\mathbf{u}) + (\nabla\mu, \nabla)\mathbf{u} - \rho\mathbf{u}_{tt} = 0 \qquad (1.1)$$

We shall assume that the Lamé parameters λ and μ and the density ρ are smooth functions of the depth z. The boundaries of the slab are free of stress

$$t_{zx} = t_{zy} = t_{zz} = 0, \qquad z = a, \qquad z = b \qquad (1.2)$$

As in [1], we examine the plane problem

$$\mathbf{u} = \mathbf{u}(x, z, t) = (u_x, u_z)$$

We seek solution of the problem (1.1) and (1.2) in the form

$$\mathbf{u}_\sigma(x, z, t, k) = e^{ikt\sigma}(G_x \sin kx, G_z \cos kx)$$

For every $\sigma \neq 0$ a solution of this form clearly is the superposition of two waves propagating along the slab in mutually opposite directions. In the sequel we shall show that such solutions exist only for certain values of σ (depending on k). For such values of σ the functions $G_x(z, k, \sigma)$ and $G(z, k, \sigma)$ decrease rapidly upon passage into the slab from the surface, which allows one to consider the corresponding solutions $\mathbf{u}_\sigma$ as surface waves.

For the vector $G(z, k, \sigma)$ we have the system

$$G'' + kAG' + k^2BG + kCG + DG' = 0 \qquad (1.3)$$

of two ordinary differential equations with smooth matrices $A(z)$, $B(z,\sigma)$, $C(z)$ and $D(z)$ and the boundary conditions

$$G' + kFG = 0, \qquad z = a, \qquad z = b \qquad (1.4)$$

In order to satisfy conditions (1.4), we shall construct the vector G in the form

$$G = \alpha_1 G^{(1)} + \alpha_2 G^{(2)} + \alpha_3 G^{(3)} + \alpha_4 G^{(4)}$$

where $G^{(j)} = G^{(j)}(z, k, \sigma)$ is one of the four linearly independent solutions of the system (1.3). The coefficients α_j are not all equal to zero when

$$\det R = 0 \qquad (1.5)$$

Here $R(k,\sigma)$ is a matrix of fourth order with the elements

$$R_{1j} = \left(\frac{d}{dz} G_x{}^{(j)} - kG_z{}^{(j)}\right)\Big|_{z=a}, \qquad R_{2j} = \left(\frac{d}{dz} G_z{}^{(j)} + k\frac{\lambda}{\nu} G_x{}^{(j)}\right)\Big|_{z=a}$$

$$R_{3j} = \left(\frac{d}{dz} G_x{}^{(j)} - kG_z{}^{(j)}\right)\Big|_{z=b}, \qquad R_{4j} = \left(\frac{d}{dz} G_z{}^{(j)} + k\frac{\lambda}{\nu} G_x{}^{(j)}\right)\Big|_{z=b}$$

$$(\nu = \lambda + 2\mu; \quad j = 1, 2, 3, 4)$$

The relation (1.5) is the dispersion equation for the phase velocity $\sigma(k)$. We shall study the solutiin of this equation for $k \gg 1$, for which we shall need an asymptotic representation of the solution of the system (1.3) for $k \to +\infty$.

The asymptotic formulas have various characteristics in cases when the interval [a,b] does or does not have a point of multiplicity (turning point). In the first case one may, as in [1], make use of the formulas of Tamarkin [2]. In the second case the asymptotics are not difficult to construct by the method of standard equations in a form adapted to a system of equations. In addition, it is possible to take advantage of the results of Feshchenko [3] and Iliukhin [4] for the "splitting" of the original system into blocks, after which the obtained block system is reduced to equations of the first and second order to which is applied the uniform asymptotic method of Langer [5] for the solution of second order equations. In this procedure, the obtaining of subsequent terms of the asymptotic series turns out to be extremely laborious. Threfore, the simplicity of the standard method and the convenience of the direct formulas that are obtained is an important advantage of this method in comparison with the method of splitting and the study of the split system.

2. Asymptotic representation of the solutions of the system (1.3). We rewrite the system (1.3) in the form of a system

$$Z' = kHZ + KZ \qquad (2.1)$$

of four ordinary differential equations of first order for the vector

$$\mathbf{Z} = (Z_1, Z_2, Z_3, Z_4) = \left(G_x, G_z, \frac{1}{k}\frac{d}{dz}G_x, \frac{1}{k}\frac{d}{dz}G_z\right)$$

The matrices H and K are as follows:

$$H = \begin{Vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ p^{-1}m_p^2 & 0 & 0 & p^{-1}-1 \\ 0 & pm_s^2 & p-1 & 0 \end{Vmatrix}, \quad K = \begin{Vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \mu'/\mu & -\mu'/\mu & 0 \\ -\lambda'/\nu & 0 & 0 & -\nu'/\nu \end{Vmatrix}$$

Here

$$m_p^2 = 1 - n_p^2(z)\sigma^2, \quad m_s^2 = 1 - n_s^2(z)\sigma^2, \quad p = \frac{v_s^2(z)}{v_p^2(z)} < \frac{1}{2}$$

$$n_p^2 = \frac{\rho(z)}{\nu(s)} = \frac{1}{v_p^2(z)}, \qquad n_s^2(z) = \frac{\rho(z)}{\mu(z)} = \frac{1}{v_s^2(z)}$$

Let us agree to call a point of multiplicity (turning point) the value of the independent variable z (depending on the parameter σ) for which two or more roots of the characteristic equation of the matrix $H(z,\sigma)$ coincide. By virtue of the inequality $v_s(z) < v_p(z)$, points of multiplicity are determined from Equations $v_s(z) = \sigma$ and $v_p(z) = \sigma$. If $v_s(z)$ and $v_p(z)$ are such that for fixed $\sigma > 0$ and $z \in [a, b]$ there are no points of multiplicity, then as is known (see for example [6]), there exists on the interval $[a,b]$ a fundamental matrix $\Phi(z,k,\sigma)$ of the system (1.3) having for $k \to \infty$ the asymptotic representation

$$\Phi(z, k, \sigma) = P\Big[\sum_{j=0}^{m-1} k^{-j} B_j(z) + O(k^{-m})\Big] \exp\Big[k\int_a^z \Lambda(z', \sigma)\, dz'\Big] \qquad (2.2)$$

Here $P(z,\sigma)$ is a matrix which reduces $H(z,\sigma)$ to the diagonal form $\Lambda = P^{-1}HP$ and the smooth matrices B_j are determined from the recursion system.

We examine the case of the presence of points of multiplicity. For simplicity we assume that $v_s(z)$ increases monotonously, whereby $v_s'(z) > 0$ on $[a,b]$ and $v_p(z) > v_s(b)$. Then it is obvious that for $v_s(a) \leqslant \sigma \leqslant v_s(b)$ there is only one point of multiplicity $z_s(\sigma) \in [a, b]$. It is not difficult to find a nonsingular transformation $\mathbf{Z} = T\mathbf{X}$ which brings H to the quasi-diagonal form

$$H^\times = T^{-1}HT = \left[\begin{Vmatrix} -m_p & 0 \\ 0 & +m_p \end{Vmatrix}, \begin{Vmatrix} 0 & 1 \\ m_s^2 & 0 \end{Vmatrix}\right]$$

The matrix T is of the form

$$T = \begin{Vmatrix} 1 & 1 & 0 & 1 \\ m_p & -m_p & -1 & 0 \\ -m_p & m_p & m_s^2 & 0 \\ -m_p^2 & -m_p^2 & 0 & -1 \end{Vmatrix}$$

For the vector $\mathbf{X}$ we have the system

$$\mathbf{X}' = kH^\times\mathbf{X} + K^\times\mathbf{X} \qquad (K^\times = T^{-1}KT - T^{-1}T') \qquad (2.3)$$

We examine the auxiliary quasi-diagonal matrix

$$W = [Y_1, Y_2]$$

Here

$$Y_1 = [J_1, J_2], \qquad J_{1,2} = \exp\left(\mp\ k\int_a^z m_p(z', \sigma)\,dz'\right)$$

$$Y_2 = \left\| \begin{matrix} y_1(z) & y_2(z) \\ k^{-1}y_1'(z) & k^{-1}y_2'(z) \end{matrix} \right\|, \qquad \begin{matrix} y_1(z) = 2k^{1/6}[\varphi'(z)]^{-1/2}v\,(k^{2/3}\varphi(z)) \\ y_2(z) = k^{1/6}[\varphi'(z)]^{-1/2}u\,(k^{2/3}\varphi(z)) \end{matrix}$$

$$\varphi(z) = \left(\frac{3}{2}\left|\int_{z_s(\sigma)}^{z} |m_s^2(z', \sigma)|^{1/2} dz'\right|\right)^{2/3} \operatorname{sgn}[z - z_s(\sigma)]$$

Here $u(t)$ and $v(t)$ are Airy functions as defined by Fok [7].

The matrix W satisfies Equation

$$W' = kH\ W + k^{-1}\,MW \tag{2.4}$$

Here

$$M(z) = [0, M_2], \qquad M_2 = \left\| \begin{matrix} 0 & 0 \\ r(z) & 0 \end{matrix} \right\|, \qquad r = \frac{\psi''}{\psi}, \qquad \psi = \frac{1}{\sqrt{\varphi'(z)}}$$

If $\lambda(z)$, $\mu(z)$ and $\rho(z)$ are infinitely differentiable then, as it is not difficult to show, one may select infinitely differentiable matrices $A_0(z)$, $A_1(z), \ldots$, such that the expression

$$X^{\vee} = \sum_{j=0}^{\infty} k^{-j} A_j(z)\, W(z)$$

will be a formal matrix solution of the system (2.3).

Indeed, the substitution of $X^{\vee}$ into the system (2.3) gives for the determination of $A_j(z)$ the recursion system

$$\begin{gathered} A_0 H^{\times} = H^{\times} A_0, \qquad A_0' + A_1 H^{\times} = H^{\times} A_1 + K^{\times} A_0 \\ A_m' + A_{m+1} H^{\times} = H^{\times} A_{m+1} + K^{\times} A_m - A_{m-1} M \qquad (m = 1, 2, \ldots) \end{gathered} \tag{2.5}$$

It is easy to verify that the general solution of the first Equations (2.5) is the matrix

$$A_0 = \left[\left\| \begin{matrix} a_0 & 0 \\ 0 & b_0 \end{matrix} \right\|, \left\| \begin{matrix} c_0 & d_0 \\ d_0 m_s^2 & c_0 \end{matrix} \right\|\right]$$

with arbitrary $a_0(z)$, $b_0(z)$, $c_0(z)$ and $d_0(z)$. We consider the second of Equations (2.5). Each matrix of fourth order we shall partition into four square blocks. The blocks will be enumerated as follows:

$$H^{\times} = \left\| \begin{matrix} H_1^{\times} & H_2^{\times} \\ H_3^{\times} & H_4^{\times} \end{matrix} \right\|, \qquad A_0 = \left\| \begin{matrix} A_{01} & A_{02} \\ A_{03} & A_{04} \end{matrix} \right\| \qquad \text{etc.}$$

For the blocks we have Equations

$$\begin{gathered} A_{01}' + A_{11} H_1^{\times} = H_1^{\times} A_{11} + K^{\times} A_{01}, \qquad A_{12} H_4^{\times} = H_1^{\times} A_{12} + K_2^{\times} A_{04} \\ A_{13} H_1^{\times} = H_4^{\times} A_{13} + K_3^{\times} A_{01}, \qquad A_{04}' + A_{14} H_4^{\times} = H_4^{\times} A_{14} + K_4^{\times} A_{04} \end{gathered} \tag{2.6}$$

The first of these equations for a_0 and b_0 gives

$$a_0 = \exp\int (K^{\times})_{11}\,dz, \qquad b_0 = \exp\int (K^{\times})_{22}\,dz$$

From this same equation, the nondiagonal elements of the matrix A_{11} are determined in a unique way by means of the already known quantities a_0 and b_0. The equation containing the matrix A_{13} uniquely determines its elements by means of $K_3^\times$ and A_{01} since $H_1^\times$ and $H_4^\times$ do not have general characteristic numbers.(*)

We turn to the equation containing A_{14}. A calculation of the matrix $K^\times$ shows that the nondiagonal elements of the block $K_4^\times$ are identically zero, which has the consequence that d_0 may be put equal to zero.

A straightforward calculation gives Formulas

$$A_{04} = [c_0, c_0], \qquad A_{14} = \left\| \begin{matrix} c_1 & d_1 \\ d_1 m_s^2 + g_1 & c_1 \end{matrix} \right\| \tag{2.7}$$
$$c_0 = \exp \frac{1}{2} \int [(K^\times)_{33} + (K^\times)_{44})]\, dz, \qquad g_1 = -c_0' + (K^\times)_{44} c_0$$

where $c_1(z)$ and $h_1(z)$ are arbitrary (until now) functions.

From equation containing A_{12}, the elements A_{12} are uniquely determined. It is essential to note that the constructed A_0 and A_1 are infinitely differentiable if H and K possess this property. This same procedure allows one to determine sequentially the remaining infinitely differentiable $A_j(z)$. Likewise, if λ, μ and ρ are smooth then, obviously, it is possible to find only a finite number of smooth A_j.

We turn to the justification of the formal expansion. It is not difficult to see that the matrix

$$X^\times = \sum_{j=0}^{m} k^{-j} A_j(z)\, W(z)$$

satisfies Equation

$$(X^\times)' = [kH^\times + K^\times + k^{-m} \Gamma(z, k, \sigma)]\, X^\times \tag{2.8}$$

where $\Gamma(z, k, \sigma) = O(1)$ for $k \to \infty$. By standard method we come to the following integral equation for the proper fundamental matrix X of the system (2.3):

$$X(z, k, \sigma) = X^\times(z, k, \sigma) - \frac{1}{k^m} \int^{(z)} X^\times(z, k, \sigma)\, [X^\times(z', k, \sigma)]^{-1}\, \Gamma(z', k, \sigma) \times \\ \times X(z', k, \sigma)\, dz' \tag{2.9}$$

We left multiply this equation by the nonsingular matrix

$$P^{-1} = \left[\sum_{j=0}^{m} k^{-j} A_j(z) \right]^{-1}$$

and introduce the notation $P^{-1}X = U$. The path of integration in Equation (2.9) can be chosen as follows

$$U(z) = W(z) + \frac{1}{k^m} \int_b^z W_1(z)\, W^{-1}(z')\, \Gamma^\times(z')\, U(z')\, dz' + \tag{2.10}$$
$$+ \frac{1}{k^m} \int_a^z W_2(z) W^{-1}(z')\, \Gamma^\times(z')\, U(z')\, dz' \qquad (\Gamma^\times = -P^{-1}\Gamma P,\ W = W_1 + W_2)$$

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*) If V_1 and V_2 are square matrices that do not have general characteristic numbers, then the matrix equation $AV_1 = V_2 A$ has the unique solution $A = 0$ (see, for example, [3]).

The partitioning of the matrix W must be different for different vectors $U^{(j)}(z)$:

$$W_1 = \left[\begin{Vmatrix} 0 & 0 \\ 0 & J_2 \end{Vmatrix}, Y_2\right] \text{ for } \mathbf{U}^{(1)} \text{ и } \mathbf{U}^{(3)}, \qquad W_1 = \left[\begin{Vmatrix} 0 & 0 \\ 0 & J_2 \end{Vmatrix}, 0\right] \text{ for } \mathbf{U}^{(2)} \text{ и } \mathbf{U}^{(4)}$$

For such a partition, as it is easy to see, the integral equation for the vector $\mathbf{U}^{(1)}J_1^{-1}$ has a bounded kernel. Indeed,

$$Y_1(z)\,Y_2^{-1}(z) = O\left(|y_1(z)\,y_2(z')| + |y_2(z)\,y_1(z')|\right)$$

But from the properties of the Airy functions (see, for example, [7]), it follows that

$$|y_1(z)\,y_2(z')| \leqslant C_1 \exp k\int_z^{z'} m_s(\zeta,\sigma)\,d\zeta \quad \text{for } z',\ z \geqslant z_s(\sigma)$$

$$|y_1(z)\,y_2(z')| \leqslant C_2 \quad \text{for } z',\ z \leqslant z_s(\sigma)$$

As a consequence of the inequality $m_p(z,\sigma) > m_s(z,\sigma)$ valid for $z \geqslant z_s(\sigma)$, the matrix $Y_2(z)\,Y_2^{-1}(z')\,J_1(z')\,J_1^{-1}(z)$ is bounded for $k \to +\infty$ on the entire square $z,\ z' \in [a, b]$. The boundedness of the first block of the matrices $W_\nu(z)\,W^{-1}(z')\,J_1(z')\,J_1^{-1}(z)$ $(\nu = 1, 2)$ is obvious.

The proof of the boundedness of the kernel of the equations for the vector $\mathbf{U}^{(2)}J_2^{-1}$. follows in an analogous way. Hence, we obtain the asymptotic formula

$$\mathbf{Z}_{(z)}{}^{(j)} = T_{(z)}\mathbf{X}_{(z)}{}^{(j)} = T\left[\sum_{\nu=0}^{m-1} k^{-\nu}A_\nu(z) + O(k^{-m})\right]\mathbf{W}_{(z)}{}^{(j)} \qquad (j = 1,\ 2) \qquad (2.11)$$

valid for $a \leqslant z \leqslant b$.

The equations for the third and fourth vectors of the matrix U have to be examined in a somewhat different way because the functions $y_1(z)$ and $y_2(z)$ have zeros for $z < z_s(\sigma)$.

If $z \geqslant z_s(\sigma)$, then by considering the equations for $\mathbf{U}^{(3)}y_1^{-1}(z)$ and $\mathbf{U}^{(4)}y_2^{-1}(z)$ we obtain a formula which is analogous to (2.11)

$$\mathbf{Z}_{(z)}{}^{(j)} = T\left[\sum_{\nu=0}^{m-1} k^{-\nu}A_\nu(z) + O(k^{-m})\right]\mathbf{W}_{(z)}{}^{(j)} \qquad (j = 3, 4) \qquad (2.12)$$

valid for $z_s(\sigma) \leqslant z \leqslant b$.

For $z \leqslant z_s(\sigma)$ the matrix $Y_2(z)$ is bounded. This leads to Formula

$$\mathbf{Z}_{(z)}{}^{(j)} = T\left[\sum_{\nu=0}^{m-1} k^{-\nu}A_\nu(z)\,\mathbf{W}^{(j)}(z) + O(k^{-m})\right] \qquad (j = 3, 4) \qquad (2.13)$$

valid for $a \leqslant z \leqslant z_s(\sigma)$.

The method which has been applied is a combination of the classical method and the method of Cherry [8]. It allows of generalization of the case of a complex region and a complex parameter. However, the choice of a path of integration is markedly more complicated in this situation as compared to the choice of the path in [8] because of the presence of the "classical" part $Y_1(z)$ of the matrix $W(z)$. It is more convenient to use the splitting method, developed for complex regions by Sibuya [9] and Wasow [10], although this method is essentially local.

3. Dispersion curves (first case). We shall construct the dispersion curves $\sigma = \sigma(k)$ for $k \gg 1$. Let $v_R(z)$ denote the velocity of a Rayleigh wave over the depth z. We shall assume that

$$v_R(a),\ v_R(b) < \min_{[a,\,b]} v_s(z) \equiv v_{sm}$$

The application of the asymptotic representation (2.2) for

$$0 < \sigma \leqslant v_{sm} - \varepsilon$$

and with $m = 1$ leads to the following representation of the dispersion equation

$$\Delta^a(k,\sigma)\,\Delta^b(k,\sigma) = S^a(k,\sigma)\,S^b(k,\sigma)\,e^{-2kf(\sigma)}\,[1 + O(e^{-kc})] \tag{3.1}$$

where

$$\Delta^z(k,\sigma) = \Delta_0^z(\sigma) + O(k^{-1}), \qquad f(\sigma) = \int_a^b m_s(z,\sigma)\,dz > 0, \qquad c > 0$$

$$S^z(k,\sigma) = S_0^z(\sigma) + O(k^{-1}),$$

$$\Delta_0^z(\sigma) = [1 + m_s^2(z,\sigma)]^2 - 4m_p(z,\sigma)\,m_s(z,\sigma)$$

$$S_0^z(\sigma) = [1 + m_s^2(z,\sigma)]^2 + 4m_p(z,\sigma)\,m_s(z,\sigma) > 0$$

It is obvious that if $v_R(a) \neq v_R(b)$ and hence $\Delta^a \neq \Delta^b$ for $k \gg 1$, Equation (3.1) has two solutions

$$\sigma_R^a(k) = \sigma_{R0}^a(k) + N^a(k)\,[1 + O(e^{-kc})]\exp[-2kf(\sigma_{R0}^a)] \tag{3.2}$$

$$\sigma_R^b(k) = \sigma_{R0}^b(k) + N^b(k)\,[1 + O(e^{-kc})]\exp[-2kf(\sigma_{R0}^b)] \tag{3.3}$$

Here $\sigma_{R0}^z(k)$ satisfies the equation $\Delta^z(k,\sigma) = 0$ and for $k \gg 1$ has the representation $\sigma_{R0}^z(k) = v_R(z) + O(k^{-1})$ (see [1]);

$$N_{(k)}^a = \left.\frac{S^a S^b}{\Delta^b \partial\Delta^a/\partial\sigma}\right|_{\sigma=\sigma_{R0}^a}, \qquad N_{(k)}^b = \left.\frac{S^a S^b}{\Delta^a \partial\Delta^b/\partial\sigma}\right|_{\sigma=\sigma_{R0}^b}$$

It is extremely essential, that by virtue of the fact that Equations (2.10) are real, Formulas (2.11) to (2.13) contain only real terms; therefore (3.2) and (3.3) are real.

If $\Delta^a(k,\sigma) \equiv \Delta^b(k,\sigma)$, then there exist two such solutions as follows:

$$\sigma_R^{\pm}(k) = \sigma_{R0}^a(k) \pm \tau(k)\cdot[1 + O(e^{-kc})]\exp[-kf(\sigma_{R0}^a)] \tag{3.4}$$

$$\left(\tau^2(k) = \left.\frac{S^a S^b}{(\partial\Delta^a/\partial\sigma)^2}\right|_{\sigma=\sigma_{R0}^a} > 0\right)$$

For $v_{sm} \leqslant \sigma \leqslant \max_{[a,b]} v_s(z)$ there exists at least one turning point $z_s(\sigma)$. We assume that $v_s'(z) > 0$. If for

$$v_{sm} + \varepsilon \leqslant \sigma \leqslant v_M - \varepsilon \qquad (v_M \equiv \min\{v_s(b),\ v_{pm}\},\ v_{pm} \equiv \min_{[a,b]} v_p(z))$$

we apply Formulas (2.11) to (2.13) and the Debye asymptotics of the Airy functions

$$u(t) = t^{-1/4}\,[1 + O(t^{-3/2})\exp {}^2/_3\, t^{3/2} \qquad \text{for } t \to +\infty$$
$$v(t) = t^{-1/4}\,[{}^1/_2 + O(t^{-3/2})]\exp(-{}^2/_3\, t^{3/2})$$

$$u(t) = \tau^{-1/4}\,[\cos({}^2/_3\tau^{3/2} + {}^1/_4\pi) + O(\tau^{-3/2})] \qquad \text{for } t = -\tau \to -\infty$$
$$v(t) = \tau^{-1/4}\,[\sin({}^2/_3\tau^{3/2} + {}^1/_4\pi) + O(\tau^{-3/2})]$$

we go over to a representation of the dispertion equation

$$\Delta_{(k,\sigma)}^{b}\Delta^{*a}(k,\sigma) = {}^1/_2 S_{(k,\sigma)}^{b} S_{(k,\sigma)}^{*a}\, e^{-2kf(\sigma)}\,[1 + O(e^{-kc})] \tag{3.5}$$

Here

$$\Delta^{*a}(k,\sigma) = \sin\Psi(k,\sigma) + O(k^{-1}), \quad S^{*a}(k,\sigma) = \cos\Psi(k,\sigma) + O(k^{-1})$$
$$\Psi(k,\sigma) = k\zeta(\sigma) + \theta(\sigma)$$
$$\theta(\sigma) = \frac{\pi}{4} - \cos^{-1}\frac{\Gamma\alpha(\sigma)}{\sqrt{\alpha^2(\sigma) + \beta^2(\sigma)}}, \qquad \zeta(\sigma) = \int_a^{z_s(\sigma)} \sqrt{\sigma^2 n_s^2(z) - 1}\, dz > 0$$
$$\alpha(\sigma) = [1 + m_s^2(a,\sigma)]^2, \qquad \beta(\sigma) = 4\,|m_s(a,\sigma)|\, m_p(a,\sigma) > 0$$

The quantities Δ^b and S^b have asymptotic representations (3.1).

Because $\sigma > v_s(a) > v_R(b)$, $\Delta^b(k,\sigma) \neq 0$, then, as is easily seen, Equation (3.5) determines a family of curves $k = k_l(\sigma)$ in the plane $k\sigma$

$$k_l(\sigma) = [l\pi - \theta(\sigma) + O(l^{-1})]\,\zeta^{-1}(\sigma) \quad (l = l_0, l_0 + 1, \ldots;\ l_0 \gg 1) \tag{3.6}$$

By virtue of $\zeta'(\sigma) > 0$, with increasing l, the modulus of inclination of this family grows without bound. The corresponding picture of the plane $k\sigma$ is schematically sketched in Fig.1.

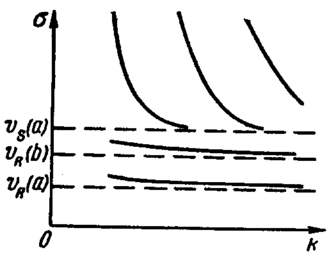

Fig. 1

If $v_{pm} > \max v_s(z)$, then on the interval $v_s(z) < \sigma < v_{pm}$ there are no turning points. In this case, Formula (2.2) leads to the asymptotic representation Δ^b and S^b of the same type as the representation Δ^{*a}, S^{*a} in Formula (3.5).

4. Dispersion curves (second case). The picture on the plane $k\sigma$ is more complicated when $v_R(b) > v_{sm}$. Again, let $v_s'(z) > 0$, and let

$$v_R(b) = v_s(z_0) < v_{pm}, \quad a < z_0 < b.$$

The application of Formulas (2.11) to (2.13) leads (again considering the region $v_{sm} < \sigma < v_M$) to Equation (3.5), but in this case $\Delta^b(k,\sigma)$ has a (simple) zero $\sigma = \sigma_{R0}^{b}(k)$. Starting from the integral equations in Section 2 and the Debye asymptotics of the Airy functions, it is not difficult to establish estimates of the type

$$k^{-1}\frac{\partial}{\partial\sigma}\,\Delta^{*a}(k,\sigma) - \zeta'(\sigma)\cos\Psi(k,\sigma) = O(k^{-1})$$

$$\frac{\partial}{\partial\sigma}\,[\Delta^{b}(k,\sigma) - \Delta_0^{b}(\sigma)] = O(k^{-1}) \qquad \text{etc.} \qquad (4.1)$$

We introduce the notation $\eta = \sigma - \sigma_{R0}^{b}(k)$. If

$$|\eta| \geqslant A_1 k^{-1-\delta}, \quad \delta > 0, \quad A_1 > 0,$$

then in this case it is obvious that

$$|\Delta^{b}(k,\sigma)| \geqslant A_2 k^{-1-\delta},$$

and once again Formula (3.6) is obtained.

Now let $\eta = O(k^{-1-\gamma})$, $\gamma > 0$. Equation (3.5) then may be rewritten in the form

$$\eta a(k,\eta)\,[b(k) + k\eta c(k,\eta)] = Q(k,\eta), \; b(k) = \Delta^{*a}(k, \sigma_{R0}^{b}(k)) \quad (4.2)$$

Here $Q(k,\eta)$ is the right-hand side of Equation (3.5) and the functions $a(k,\eta)$ and $c(k,\eta)$ have the following asymptotic representations:

$$a(k,\eta) = \frac{d}{d\sigma}\,\Delta_0^{b}\big|_{\sigma=v_R(b)} + O(k^{-1}) > 0$$

$$c(k,\eta) = k^{-1}\frac{\partial}{\partial\sigma}\,\Delta^{*a}(k,\sigma)\big|_{\sigma=\sigma_{R0}^{b}} + O(k^{-1}) \qquad (4.3)$$

A straightforward examination of Equation (4.2) shows that the presence of the right-hand side $Q = O(e^{-2kf})$ essentially affects the behavior of the curves $\sigma = \sigma_i(k)$ in the neighborhood of the curve $\sigma = \sigma_{R0}^{b}(k)$. Namely, if $Q \equiv 0$, we would have as a solution of Equation (4.2) the family (3.6) and the curve $\sigma = \sigma_{R0}^{b}(k)$. which intersects this family. As should be expected, the presence of Q leads to the disappearance of the intersection points of the dispersion curves. As it is easy to calculate, the neighboring curves have their places of nearest approach at the points

$$k_m = [m\pi - (\theta + v_1\zeta')\,|_{\sigma=v_R(b)} + O(m^{-1})]\;\zeta^{-1}(v_R(b)) \qquad (4.4)$$

$$v_1 = \lim\,[\sigma_{R0}^{b} - v_R(b)]\,k \quad \text{for } k\to\infty \qquad (m = m_0,\, m_0+1, \ldots;\; m_0 \gg 1)$$

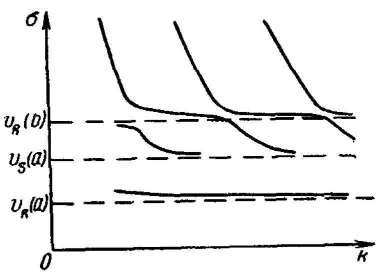

Fig. 2

The distance in the vertical direction between neighboring curves turns out in this case to be equal to

$$Bk^{-1/2}[1 + O(k^{-1})]\exp[-kf(v_R(b))] \quad (4.5)$$

$$\left(B^2 = \frac{S_0^{b} e^{-v_1 f'}}{2\zeta' d\Delta_0^{b}/d\sigma}\bigg|_{\sigma=v_R(b)} > 0 \right)$$

The properties of the dispersion curves for $\sigma \approx v_{s}$ may be investigated by the application of the same Formulas (2.11) to (2.13). A schematic picture of the distribution of the curves in the $k\sigma$ plane is given in Fig.2.

The region $\sigma > v_{p*}$ may be investigated in a completely analogous way to the above; in which case it is clear that formulas of the type (2.11) to (2.13) will contain the point $z_p(\sigma)$ as a turning point.

In conclusion we note that by patching together formulas of type (2.11) to (2.13) which are suitable to intervals of monotone velocity, it is possible to study Rayleigh waves in a medium with nonmonotone propagation velocities.

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